

On the use of Helmholtz resonators as sound attenuators

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1. Introduction

The problem of nonlinear Helmholtz resonators has been addressed by many previous investigators. The character of the nonlinearity and various additional aspects of resonant and nearly resonant oscillations have been discussed by Ingard [1], Bies and Wilson [2], Gill and Bhaduri [3], Barthel [4], Borisova et al. [5] and Ochmann [6]. However, there still doesn't seem to exist a systematic investigation of the relative importance of a variety of effects, including those of thermoacoustic Stokes boundary layers at the walls of the throats, a non-zero mean flow through the resonator, the conductance of the ends of the throats and higher-order frequency corrections to the "Helmholtz approximation".

In the limit of the Helmholtz approximation spatial phase variations within the individual elements (throats and volumes) of a resonator become negligibly small. In this limit a resonant oscillation may be regarded as a periodic exchange process that involves conversion of potential energy from pressurized volume elements into kinetic energy of moving gas columns in throat elements, and vice versa. Due to radiation of sound, boundary layer dissipation and internal jet dissipation a (small) fraction of the acoustic energy, stored in a resonator, gets lost per cycle and must be compensated by an excitation mechanism to sustain an equilibrium oscillation. Typically such an excitation is caused by a pressure oscillation at an outer end of a throat element. To obtain an accurate prediction of oscillation amplitudes all the aforementioned mechanisms of excitation and attenuation must usually be taken into account. Furthermore, lowest-order theories of nonlinear acoustic oscillations typically predict the resonance frequencies to first-order accuracy only. Higher-order frequency corrections are usually

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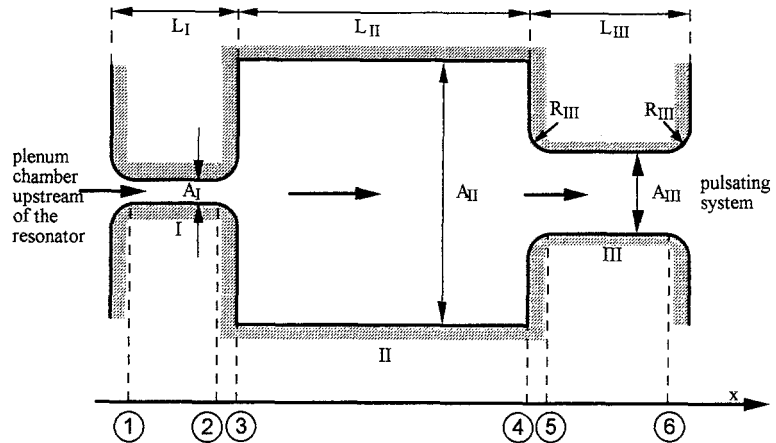


Figure 1.1
Arrangement of a Helmholtz resonator with two throats and flushing.

taken into account, because most technical applications require knowledge of resonance frequencies that is of better accuracy than first order.

For some technical applications a principal additional difficulty arises due to the necessity of permanently flushing the resonator, in order to avoid an overheating produced by the radiation of hot gas. The idea is to develop a design method which leads to reliable predictions of the resonance frequency, the width of the resonance band and the attenuation power of a resonator element. Although, the width of the resonance band can be changed by changing the geometry of the Helmholtz resonator, it is clear that the relative width of the resonance band is always fairly small. For this reason Helmholtz resonators are very effective attenuators for narrow frequency intervals. However, they are not at all suitable for broad band sound attenuation. For a variety of configurations and conditions the properties of Helmholtz resonators have been investigated with the help of the experimental results reported in this paper. It will be seen that the effects of throughflow, the geometry of the ends of throats, higher order corrections of the Helmholtz approximation and Stokes boundary layers in the throats are usually of fundamental importance for an accurate theoretical prediction of the properties of a Helmholtz resonator.

The situation presently considered is depicted in Fig. 1.1.

2. Resonance frequency in the limit of negligibly small boundary layer friction

The Helmholtz resonator shown in Fig. 1.1 consists of the three tube elements designated by *I*, *II* and *III*. Assuming that the diameters and lengths of the tube elements are very small compared to the relevant

acoustic wave lengths and that the attenuation of acoustic waves within the tube elements due to boundary layer friction is negligibly small, the downstream and upstream running acoustic waves (i.e. Riemann invariants) are defined by the expressions

$$f \equiv \Delta u + \frac{\Delta p}{\rho c}, \quad g \equiv \Delta u - \frac{\Delta p}{\rho c}, \quad (2.1)$$

where Δu and Δp denote velocity and pressure disturbance, ρ refers to the density and c to the sound speed. All quantities which are constant within the tube elements of the Helmholtz resonator are marked with suffices “I”, “II”, “III” respectively. Hence, the mean flow Mach number is denoted by M_I , for example. The discussion in this section is restricted to the small amplitude limit of acoustic oscillations. Hence, linear acoustic theory is the adequate tool for the subsequent consideration. The reflection condition at the inlet of the first tube of the resonator can be defined by

$$f_1 = \frac{1 - \zeta_I M_I}{1 + \zeta_I M_I} g_1, \quad (2.2)$$

where “1” refers to the location in the resonator (see Fig. 1.1) and ζ is the loss coefficient of the tube inlet; $\zeta_I = 1$ would correspond to a loss free inlet with a static pressure drop of *one* dynamic head across the tube inlet.

As only the relative amplitudes of the acoustic waves are relevant for the present linearized analysis, we introduce the normalization

$$f_1 = 1 - \zeta_I M_I \quad \text{and} \quad g_1 = 1 + \zeta_I M_I. \quad (2.3)$$

Harmonic Riemann invariants f and g at the locations 1 to 6 are related by the expressions

$$\begin{bmatrix} f_2 \\ g_2 \end{bmatrix} = \begin{bmatrix} \exp\left(\frac{-i\omega L_I}{(1 + M_I) \cdot c}\right) & 0 \\ 0 & \exp\left(\frac{i\omega L_I}{(1 - M_I) \cdot c}\right) \end{bmatrix} * \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \quad (2.4)$$

$$\begin{bmatrix} f_3 \\ g_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \left(1 + \frac{A_I}{A_{II}}\right) & -\frac{1}{2} \left(1 - \frac{A_I}{A_{II}}\right) \\ -\frac{1}{2} \left(1 - \frac{A_I}{A_{II}}\right) & \frac{1}{2} \left(1 + \frac{A_I}{A_{II}}\right) \end{bmatrix} * \begin{bmatrix} f_2 \\ g_2 \end{bmatrix}, \quad (2.5)$$

$$\begin{bmatrix} f_4 \\ g_4 \end{bmatrix} = \begin{bmatrix} \exp\left(\frac{-i\omega L_{II}}{(1 + M_{II}) \cdot c}\right) & 0 \\ 0 & \exp\left(\frac{i\omega L_{II}}{(1 - M_{II}) \cdot c}\right) \end{bmatrix} * \begin{bmatrix} f_3 \\ g_3 \end{bmatrix}, \quad (2.6)$$

$$\begin{bmatrix} f_5 \\ g_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \left[(1 - \zeta_{III} M_{III}) \frac{A_{II}}{A_{III}} + 1 \right] & + \frac{1}{2} \left[(1 - \zeta_{III} M_{III}) \frac{A_{II}}{A_{III}} - 1 \right] \\ \frac{1}{2} \left[(1 + \zeta_{III} M_{III}) \frac{A_{II}}{A_{III}} - 1 \right] & + \frac{1}{2} \left[(1 + \zeta_{III} M_{III}) \frac{A_{II}}{A_{III}} + 1 \right] \end{bmatrix} * \begin{bmatrix} f_4 \\ g_4 \end{bmatrix}, \quad (2.7)$$

$$\begin{bmatrix} f_6 \\ g_6 \end{bmatrix} = \begin{bmatrix} \exp\left(\frac{-i\omega L_{III}}{(1 + M_{III}) \cdot c}\right) & 0 \\ 0 & \exp\left(\frac{i\omega L_{III}}{(1 - M_{III}) \cdot c}\right) \end{bmatrix} * \begin{bmatrix} f_5 \\ g_5 \end{bmatrix}, \quad (2.8)$$

where A , L and ω denote the cross-sectional area of a tube element, its length and the angular frequency of the harmonic waves respectively.

While the phase shifts defined by the relations (2.4), (2.6) and (2.8) are obviously related to the lengths of the tube elements and the mean flow speed $M \cdot c$, the relation (2.5) is obtained on the basis of the assumptions that the static pressures at the locations 2 and 3 are equal (fully separated free jet) and that the phase shift between these two locations is negligibly small. The latter assumption also holds for the relation (2.7). However, the static pressure drop between the locations 4 and 5 is taken into account. For a loss-free flow between these two locations the coefficient ζ_{III} would assume the value

$$\zeta_{III} = 1 - \left(\frac{A_{III}}{A_{II}} \right)^2. \quad (2.9)$$

To obtain the pressure oscillation at location 6 we evaluate the term $\frac{1}{2}(f_6 - g_6)$. Within the "Helmholtz approximation", which is discussed subsequently, the Mach number corrections contained in the phase exponents of (2.4), (2.6) and (2.8) can be neglected. Combining the relations (2.3) to (2.8) and neglecting the Mach number corrections in (2.4), (2.6) and (2.8) leads to the expression (A1) given in the Appendix.

Now we restrict our consideration to the lowest resonant frequency and make use of the "Helmholtz approximation". In this case the expression (A1) can be simplified by introducing the following hierarchy of orders:

$$\begin{aligned} \varepsilon &\equiv O\left(\frac{\omega L_I}{c}\right) = O\left(\frac{\omega L_{II}}{c}\right) = O\left(\frac{\omega L_{III}}{c}\right) \ll 1, \\ O\left(\frac{A_I}{A_{II}}\right) &= O\left(\frac{A_{III}}{A_{II}}\right) = \varepsilon^2 \end{aligned} \quad (2.10)$$

and

$$O(M_I) = O(M_{III}) \leq \varepsilon. \quad (2.11)$$

For a given amplitude of the pressure oscillation at location 6 a necessary condition for resonance is that the amplitude of the pressure oscillation in the vessel *II* reaches a local maximum. However, because we have already fixed the amplitudes of the waves by the normalization (2.3) we have to keep the amplitude of the pressure oscillation at location 6 variable. *For this reason the corresponding necessary condition for resonance is that the amplitude of the pressure oscillation at location 6 reaches a local minimum.*

Making use of the substitution

$$\frac{1}{2}(f_6 - g_6) = \xi + i\eta, \quad \text{where } \xi \text{ and } \eta \text{ are real,}$$

this condition can be written in the form

$$\frac{\partial}{\partial \omega}(\xi^2 + \eta^2) = 0. \quad (2.12)$$

Inserting the expression (A1) into the condition (2.12) and making use of the hierarchy of orders as defined by (2.10) and (2.11) leads to lowest order to (A2) as given in the Appendix.

In the limit $M_I, M_{III} \rightarrow 0$, the resonance condition reduces to the condition that the underlined term within the first square bracket of (A2) vanishes. Thus we have

$$\frac{\omega^2}{c^2} = \frac{1}{V} \cdot \left[\frac{A_I}{L_I} + \frac{A_{III}}{L_{III}} \right], \quad \text{where } V = L_{II} \cdot A_{II}, \quad (2.13)$$

which represents the well-known Helmholtz formula for a resonator with two throats.

From a practical point of view the following special case is particularly important

$$O\left(\frac{A_I}{A_{III}}\right) = O\left(\frac{M_{III}}{M_I}\right) \ll 1. \quad (2.14)$$

It is quite remarkable that in this case the general condition (A2) reduces to

$$\frac{\omega^2}{c^2} = \frac{A_{III}}{VL_{III}}, \quad (2.15)$$

which is the simple Helmholtz formula for a resonator with one throat only.

In the general case where no further simplifications are justified the resonance frequency is obtained by solving equation (A2) for $(\omega/c)^2$.

3. The attenuation of waves due to the thermoacoustic boundary layer

Assuming that the mean velocity in the tube elements (i.e. the throats *I* and *III*) is very small and that the thicknesses of the boundary layers are

small compared to the tube radii, leads to an exponential attenuation of harmonic waves:

$$f = a \cdot \exp\left(i\omega t - \frac{i\omega x}{c} - \alpha x\right)$$

and (3.1)

$$g = b \cdot \exp\left(i\omega t + \frac{i\omega x}{c} + \alpha x\right),$$

where $|\alpha| \ll \omega/c$.

Following Rott and Hartunian [7] (see also Chester [8] and Keller [9]) we find

$$\alpha = [1 + i] \cdot \frac{\sqrt{2\omega\nu}}{D \cdot c} \cdot \left[1 + \frac{\gamma - 1}{\text{Pr}^{1/2}}\right], \quad (3.2)$$

where ν denotes the kinematic viscosity of the air, D the diameter of the tube, γ the ratio of specific heats and Pr the Prandtl number.

Hence, the attenuation factor of a harmonic wave running from one end of a throat of length L and cross-sectional area A to the other is

$$\exp[-\text{Re}(\alpha) \cdot L] = \exp\left\{-\sqrt{\frac{\omega \cdot \nu \cdot \pi}{2 \cdot A}} \cdot \frac{L}{c} \cdot \left[1 + \frac{\gamma - 1}{\text{Pr}^{1/2}}\right]\right\}, \quad (3.3)$$

where $\text{Re}(\alpha)$ refers to the real part of α .

As long as the mean flow speed in a tube does not exceed the order of magnitude of the amplitude of the velocity oscillation, the viscous effects due to the oscillatory part of the velocity are not affected to first approximation by the mean flow speed.

Hence, we can restrict our discussion to the case of zero mean flow speed. To fix ideas we consider a pair of upstream and downstream running waves of the form (3.1) in a tube of length L and diameter D . Furthermore, we set the origin of the coordinate system such that $x = -\frac{1}{2}L$ and $x = +\frac{1}{2}L$ correspond to the two ends of the tube (see Fig. 3.1).

For reasons of symmetry we can now set $b = a$.

Combining (2.1) and (3.1) yields

$$\Delta u = \frac{1}{2}(f + g) = a \cdot e^{i\omega t} \cdot \cosh\left(\frac{i\omega x}{c} + \alpha x\right) \quad (3.4)$$

and

$$\frac{\Delta p}{\rho c} = \frac{1}{2}(f - g) = -a \cdot e^{i\omega t} \cdot \sinh\left(\frac{i\omega x}{c} + \alpha x\right). \quad (3.5)$$

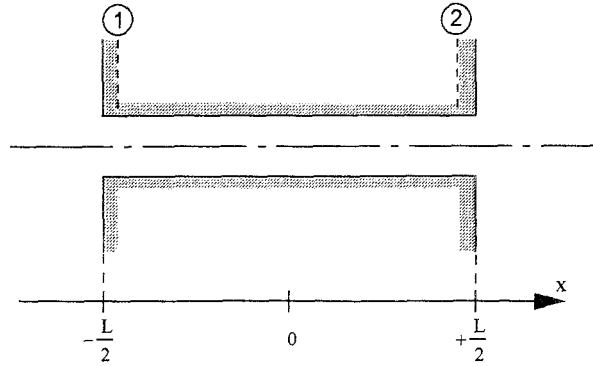


Figure 3.1
Coordinate system in a tube element.

Making use of the Helmholtz approximation as defined by (2.10), the expressions (3.4) and (3.5) can be simplified:

$$\Delta u = a \cdot e^{i\omega t}, \quad \frac{\Delta p}{\rho c} = -a \cdot e^{i\omega t} \cdot \left[\frac{i\omega x}{c} + \alpha x \right]. \quad (3.6)$$

Hence, we obtain (see locations marked in Fig. 3.1)

$$\Delta p_1 - \Delta p_2 = a \cdot e^{i\omega t} \cdot \left[\frac{i\omega L}{c} + \alpha L \right] \cdot \rho c. \quad (3.7)$$

Introducing now the boundary layer parameter

$$s = \frac{1 + \frac{\gamma - 1}{\text{Pr}^{1/2}}}{D} \cdot \sqrt{\frac{2\nu}{\omega}} \quad (3.8)$$

the equations (3.2), (3.6) and (3.7) yield

$$\Delta p_1 - \Delta p_2 = [1 + s] \cdot \rho \cdot L \cdot \frac{\partial}{\partial t} \Delta u + s \cdot \rho \cdot L \cdot \omega \cdot \Delta u. \quad (3.9)$$

The first term on the right hand side of equation (3.9) obviously corresponds to the inertia of the gas column in the tube, including a correction due to the boundary layer, and it is non-dissipative, whereas the second term accounts for the viscous effect on the pressure difference along the tube.

4. The Helmholtz approximation of the equations of motion

We return now to consider the complete resonator as shown in Fig. 1.1. Within the Helmholtz approximation, which was already discussed at the

end of Section 2, the condition for mass conservation can be expressed as follows:

$$\dot{q}V = (U_I + u_I) \cdot q \cdot A_I - (U_{III} + u_{III}) \cdot q \cdot A_{III}, \quad (4.1)$$

where U_I and U_{III} denote the mean values of the velocities in the tube elements I and III , respectively, and u_I and u_{III} refer to the departures of the velocities from their mean values. Within the same approximation (4.1) implies that

$$U_I A_I - U_{III} A_{III} = 0 \quad (4.2)$$

and

$$\dot{q}V = u_I \cdot q \cdot A_I - u_{III} \cdot q \cdot A_{III}. \quad (4.3)$$

Denoting the pressure in the pulsating system near the end of the tube element III by p_c and the pressure in the resonator vessel II by p_v and ignoring in a first step the boundary layer effects, the equation of motion of the gas column in the tube element III can be written in the form

$$A_{III} L_{III} \dot{q} u_{III} = A_{III} p_v - A_{III} \cdot \zeta_{III} \cdot \frac{q}{2} \cdot (U_{III} + u_{III}) \cdot |U_{III} + u_{III}| \cdot - A_{III} \cdot p_c, \quad (4.4)$$

where a dot above a variable refers to a time derivative.

The term on the left-hand side of equation (4.4) represents the force due to the acceleration of the gas column, the first and third terms on the right-hand side account for the force due to the different pressures in the resonator vessel and the pulsating system, and the second term on the right-hand side of equation (4.4) accounts for the force due to the acceleration of fluid from the resonator vessel into the tube element III or from the pulsating system into the tube element III , depending on the sign of the velocity ($U_{III} + u_{III}$). In the case of loss-free acceleration ζ_{III} is defined by (2.9) for positive velocities and $\zeta_{III} \approx 1$ for negative velocities. For simplicity we assume that even for negative velocities hot fluid does not penetrate from the pulsating system into the tube element III . Hence, the density q does not change within the approximation considered. Here we assume that ζ_{III} assumes the same value for positive and negative velocities. However, a more detailed investigation shows that the general form of the second term on the right-hand side of equation (4.4) remains unchanged even for a strong dependence of ζ_{III} on the sign of the velocity, provided a loss coefficient is defined which corresponds to an average over an oscillation period. It should be pointed out that such an average strongly depends on the relative magnitudes of U_{III} and u_{III} . In situations of practical interest, in which the Helmholtz resonator is used as an attenuator, the aim is to keep ζ_{III} as small as possible. In this case ζ_{III} would usually not depend on the

sign of the velocity ($U_{III} + u_{III}$), because the two ends of the tube element *III* would typically exhibit the same geometry.

On the basis of the Helmholtz approximation (see (2.10) and (2.11)) it is easy to show that

$$O(\dot{q}\dot{u}_{III}) \ll (q\ddot{u}_{III}) \quad (4.5)$$

and

$$O(\dot{q} \cdot [U_{III} + u_{III}]) \ll O(q\dot{u}_{III}). \quad (4.6)$$

Differentiating (4.4) with respect to time and making use of (4.5) and (4.6) yields

$$L_{III} \cdot q \cdot \ddot{u}_{III} = \dot{p}_v - \zeta_{III} \cdot q \cdot \dot{u}_{III} \cdot |U_{III} + u_{III}| - \dot{p}_c. \quad (4.7)$$

It should be pointed out that a mean-flow pressure drop in the tube element, which is produced by boundary layer friction, can be included by means of a suitable correction of ζ_{III} . However, the effects produced by a Stokes boundary layer are different. A more general version of the equation of motion, which does account for the Stokes boundary layer, is obtained by including the corresponding terms according to (3.9) in equation (4.7):

$$\begin{aligned} [1 + s_{III}] \cdot L_{III} \cdot q \cdot \ddot{u}_{III} = & \dot{p}_v - \zeta_{III} \cdot q \cdot \dot{u}_{III} \cdot |U_{III} + u_{III}| \\ & - s_{III} \cdot L_{III} \cdot q \cdot \omega \cdot \dot{u}_{III} - \dot{p}_c, \end{aligned} \quad (4.8)$$

where

$$s_{III} = \left[1 + \frac{\gamma - 1}{\text{Pr}^{1/2}} \right] \cdot \sqrt{\frac{\pi \cdot \nu}{2 \cdot \omega \cdot A_{III}}}. \quad (4.9)$$

For tube elements with non-circular cross-sections the diameter $\sqrt{4A_{III}/\pi}$ in (4.9) should be replaced by the hydraulic diameter.

For the subsequent discussion we assume that the time-dependent part of the pressure in the pulsating system (near the end of the tube element *III*) is a harmonic oscillation, i.e.

$$p_c = \langle p_c \rangle - \hat{p} \cdot \sin(\omega \cdot [t - t_0]), \quad (4.10)$$

where “ $\langle \rangle$ ” refers to a time average, $\hat{p}$ is an oscillation amplitude and t_0 accounts for a shift of the origin of the time coordinate which will be chosen suitably below. Furthermore, we assume that the pressure fluctuation in the chamber upstream of the resonator is negligibly small. Inserting (4.10) in (4.8) yields

$$\begin{aligned} [1 + s_{III}] \cdot L_{III} \cdot q \cdot \ddot{u}_{III} = & \dot{p}_v - \zeta_{III} \cdot q \cdot \dot{u}_{III} \cdot |U_{III} + u_{III}| - s_{III} \cdot L_{III} \\ & \cdot q \cdot \omega \cdot \dot{u}_{III} + \omega \cdot \hat{p} \cdot \cos(\omega \cdot [t - t_0]). \end{aligned} \quad (4.11)$$

A corresponding consideration leads to the equation of motion for the tube element I :

$$[1 + s_I] \cdot L_I \cdot \varrho \cdot \ddot{u}_I = -\zeta_I \cdot \varrho \cdot \dot{u}_I \cdot |U_I + u_I| - s_I \cdot L_I \cdot \varrho \cdot \omega \cdot \dot{u}_I - \dot{p}_v. \quad (4.12)$$

Within the present approximation it is justified to use the following relation (which is strictly valid for isentropic flow):

$$\dot{p}_v = c^2 \dot{\varrho}. \quad (4.13)$$

Combining (4.3) and (4.13) leads to

$$\dot{p}_v = \frac{A_I u_I \varrho c^2}{V} - \frac{A_{III} u_{III} \varrho c^2}{V}. \quad (4.14)$$

As will be shown subsequently, a harmonic pressure oscillation of the pulsating system also leads to harmonic velocity oscillations in the tube elements and a harmonic pressure oscillation in the resonator vessel within the approximation considered.

The time shift t_0 appearing in the expression (4.10) is now implicitly determined by choosing the following expression for the velocity oscillation in the tube element III :

$$u_{III} = \hat{u}_{III} \cdot \sin(\omega \cdot t). \quad (4.15)$$

For the velocity oscillation in the tube element I we write

$$u_I = -\hat{u}_I \cdot \sin(\omega \cdot [t - \tau]). \quad (4.16)$$

The symbol “ $\wedge$ ” again refers to amplitudes, and the phase shift associated with the constant τ will have to be determined subsequently.

Inserting (3.14) in (4.11) and (4.12), in order to eliminate p_v , leads to

$$\begin{aligned} (1 + s_{III}) \cdot L_{III} \cdot \varrho \cdot \ddot{u}_{III} + \frac{A_{III} \cdot \varrho \cdot c^2}{V} \cdot u_{III} - \frac{A_I \cdot \varrho \cdot c^2}{V} \cdot u_I \\ = -\zeta_{III} \cdot \varrho \cdot \dot{u}_{III} \cdot |U_{III} + u_{III}| - s_{III} \cdot L_{III} \cdot \varrho \cdot \omega \cdot \dot{u}_{III} \\ + \omega \cdot \hat{p} \cdot \cos(\omega \cdot [t - t_0]) \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} (1 + s_I) \cdot L_I \cdot \varrho \cdot \ddot{u}_I + \frac{A_I \cdot \varrho \cdot c^2}{V} \cdot u_I - \frac{A_{III} \cdot \varrho \cdot c^2}{V} \cdot u_{III} \\ = -\zeta_I \cdot \varrho \cdot \dot{u}_I \cdot |U_I + u_I| - s_I \cdot L_I \cdot \varrho \cdot \omega \cdot \dot{u}_I. \end{aligned} \quad (4.18)$$

5. Properties of the oscillation equations at and near resonance

With the equations (4.17) and (4.18) we have now obtained a complete system to describe small amplitude nonlinear oscillations of the resonator shown in Fig. 1.1 at and near resonance.

If the right-hand sides of these equations were equal to zero, the system would become linear, and it would describe a neutrally stable coupled oscillation. The third term on the right-hand side of equation (4.17) accounts for the excitation, the first terms on the right-hand sides of the equations (4.17) and (4.18) describe the nonlinear attenuation due to free jet dissipation and the second terms account for the linear attenuation due to the Stokes boundary layer.

Special case: the simple Helmholtz resonator

In order to simplify the oscillation equations we first need to study certain basic properties of the equations. For this purpose we do not need to consider the relatively complicated general case. It is sufficient to investigate the envisaged properties in the case of the simple Helmholtz resonator as illustrated in Fig. 5.1.

Setting $A_I = 0$, $t_0 = 0$, $U_I = U_{III} = 0$ and dropping the suffix “III”, the system ((4.17), (4.18)) is reduced to a single equation:

$$(1 + s) \cdot L \cdot \varrho \cdot \ddot{u} + \frac{A \cdot \varrho \cdot c^2}{V} \cdot u = -\zeta \cdot \varrho \cdot \dot{u} \cdot |u| - s \cdot L \cdot \varrho \cdot \omega \cdot \dot{u} + \omega \cdot \hat{p} \cdot \cos(\omega \cdot t). \quad (5.1)$$

This equation describes the oscillation of an ordinary Helmholtz resonator without throughflow. For a harmonic excitation as described by the third term on the right-hand side of (5.1), the solution u can be written as a

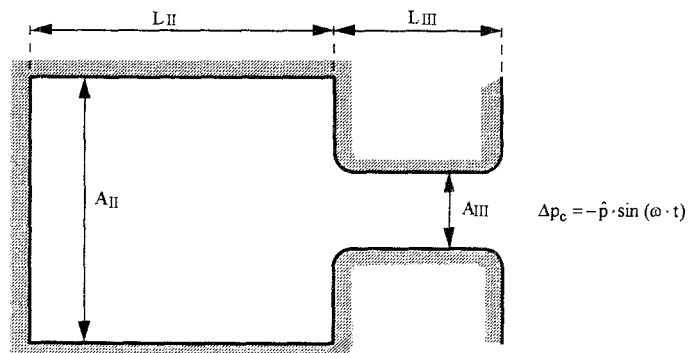


Figure 5.1
Helmholtz resonator with one throat only (closed resonator).

Fourier series:

$$u = \sum_{n=1}^{\infty} [a_n \cdot \cos(\omega t) + b_n \cdot \sin(\omega t)], \quad (5.2)$$

where a_n and b_n are constants. If the right-hand side of (5.1) were equal to zero the oscillation would again become neutrally stable. The corresponding resonant frequency is

$$\omega_0 = c \cdot \left[\frac{A}{(1+s) \cdot V \cdot L} \right]^{1/2}. \quad (5.3)$$

Hence, at resonance (i.e. for $\omega = \omega_0$)

$$(1+s) \cdot L \cdot \ddot{u}_1 + \frac{Ac^2}{V} \cdot u_1 = 0, \quad (5.4)$$

where

$$u_1 = a_1 \cdot \cos(\omega \cdot t) + b_1 \cdot \sin(\omega \cdot t). \quad (5.5)$$

For this reason the third term on the right-hand side of equation (5.1) has to be compensated by the first and second terms on the right-hand side of (5.1) alone, if $\omega = \omega_0$. Thus, if the effects due to the Stokes boundary layer are very small, we have

$$\max(O(a_1), O(b_1)) = \left(\frac{\hat{p}}{\varrho \zeta} \right)^{1/2}. \quad (5.6)$$

On the other hand, if the attenuation due to the Stokes boundary layer is much stronger than the nonlinear attenuation, we obtain

$$\max(O(a_1), O(b_1)) = \left(\frac{\hat{p}}{s \cdot L \cdot \varrho \cdot \omega} \right). \quad (5.7)$$

In the latter case it is obvious that the higher harmonics in (5.2) are very weak compared to the fundamental harmonic. However, it can also be shown that the same is true for weak effects of the Stokes boundary layer: Inserting (5.2) in (5.1) and ignoring the effects due to the Stokes boundary layer (i.e. $s \approx 0$) we obtain

$$\max(O(a_2), O(b_2)) = \sqrt{\frac{V}{AL}} \cdot \frac{\hat{p}}{\varrho c}. \quad (5.8)$$

Hence,

$$\max(O(a_2), O(b_2)) \ll \max(O(a_1), O(b_1)), \quad (5.9)$$

if

$$\frac{\hat{p}}{p_c} \ll \frac{\gamma}{\zeta} \cdot \frac{AL}{V}. \quad (5.10)$$

In a situation of practical interest the left-hand side of (5.10) would typically be two orders of magnitude smaller than the right-hand side. Similarly, it can be shown that the complete departure of u from its fundamental harmonic remains small if (5.10) applies, i.e.

$$O\left(\sum_{n=2}^{\infty} [a_n \cdot \cos(\omega \cdot t) + b_n \cdot \sin(\omega \cdot t)]\right) \ll O(u). \quad (5.11)$$

As the amplitude of u decreases rapidly as $|\omega - \omega_0|$ increases, and the first term on the right-hand side of (5.1) becomes very small compared to the left-hand side of (5.1), the relation (5.11) remains valid for $\omega \neq \omega_0$.

To summarize the discussion we may conclude that the solution u of equation (5.1) is nearly sinusoidal at and near resonance, provided the amplitudes of the pressure oscillations remain small according to the criterion (5.10).

General case

For rather obvious reasons this general statement remains to be true if we extend the discussion again from the oscillation in an ordinary Helmholtz resonator as described by equation (5.1) to the more general case discussed in Section 4. Thus, we return to the discussion of the set of equations (4.17) and (4.18). In the light of the above discussion it is justified to replace the nonlinear terms in (4.17) and (4.18) (i.e. the first terms on the right-hand sides) by their fundamental harmonics.

A Fourier analysis shows that if u_{III} is sinusoidal, as defined by (4.15), the fundamental harmonic of the nonlinear term

$$-\zeta_{III} \cdot \varrho \cdot \dot{u}_{III} \cdot |U_{III} + u_{III}|$$

is

$$-\zeta_{III} \cdot \varrho \cdot \omega \cdot \hat{u}_{III}^2 \cdot g\left(\frac{U_{III}}{\hat{u}_{III}}\right) \cdot \cos(\omega \cdot t), \quad (5.12)$$

where

$$g(x) = \begin{cases} \frac{1}{\pi} \cdot \left[2 \cdot x \cdot \arcsin(x) + \frac{2}{3} \sqrt{1-x^2} \cdot (2+x^2) \right], & \text{if } 0 \leq x \leq 1 \\ x, & \text{if } 1 < x. \end{cases} \quad (5.13)$$

The graph of the function g is shown in Fig. 5.2.

Replacing now the first term on the right-hand side of equation (4.17) by the expression (5.12) and the first term on the right-hand side of equation (4.18) by the corresponding fundamental harmonic leads to the

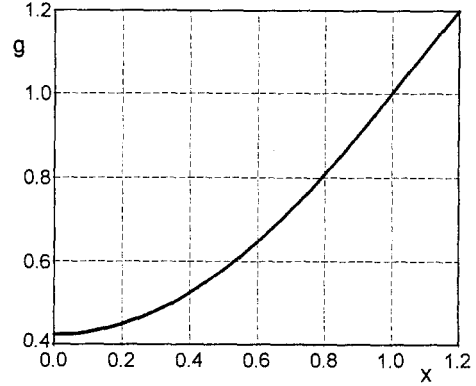


Figure 5.2
Function g representing the fundamental harmonic
of the nonlinear term.

following simplified equations for the fundamental harmonics:

$$\begin{aligned}
 & (1 + s_{III}) \cdot L_{III} \cdot \ddot{u}_{III} + \frac{A_{III} \cdot c^2}{V} \cdot u_{III} - \frac{A_I \cdot c^2}{V} \cdot u_I \\
 &= \frac{\omega \cdot \hat{p}}{\varrho} \cdot \cos(\omega \cdot [t - t_0]) - \zeta_{III} \cdot \omega \cdot \hat{u}_{III}^2 \cdot g\left(\frac{U_{III}}{\hat{u}_{III}}\right) \cdot \cos(\omega \cdot t) \\
 & \quad - s_{III} \cdot L_{III} \cdot \omega \cdot \dot{u}_{III}, \tag{5.14}
 \end{aligned}$$

$$\begin{aligned}
 & (1 + s_I) \cdot L_I \cdot \ddot{u}_I + \frac{A_I \cdot c^2}{V} \cdot u_I - \frac{A_{III} \cdot c^2}{V} \cdot u_{III} \\
 &= \zeta_I \cdot \omega \cdot \hat{u}_I^2 \cdot g\left(\frac{U_I}{\hat{u}_I}\right) \cdot \cos(\omega \cdot [t - \tau]) - s_I \cdot L_I \cdot \omega \cdot \dot{u}_I. \tag{5.15}
 \end{aligned}$$

A procedure to solve the system of equations (5.15) and (5.16) is given in the Appendix.

The lengths of the tube elements (particularly L_{III}) used in the relations (A3) to (A6) should not correspond to the geometric lengths, but to the geometric lengths including a length correction due to the particular impedances of the tube ends (see Morse and Ingard [10], section 9.1). For an unflanged open end, for example, the length correction corresponds to about 30% of the tube diameter (see Levine and Schwinger [11]). A further virtual increase of the tube lengths is due to higher orders of $\omega L_I/c$, $\omega L_{II}/c$ and $\omega L_{III}/c$ in (A1) which are ignored within the Helmholtz approximation. An analysis which includes these nonlinear terms leads to the “effective length”

$$L_{III_{EFF}} = L_{III} \cdot \left[1 + \frac{1}{3} \cdot \left(\frac{\omega}{c} \right)^2 \cdot (L_{II}^2 + L_{III}^2) \right] \tag{5.16}$$

for the situation illustrated by Fig. 5.1, where L_{III} is the length which already includes the “impedance correction” and $L_{III_{EFF}}$ also includes the first correction to the Helmholtz approximation.

6. The solutions and their properties

Increasing the mass flow through a Helmholtz attenuator as shown in Fig. 1.1 (and keeping the amplitude $\hat{p}$ of the pressure oscillation in the pulsating system constant) leads to a reduction of the velocity amplitude $\hat{u}_{III}$ and consequently also to a reduction of the sound attenuation. Furthermore, the resonance frequency is reduced and the width of the resonance band increased.

Table 6.1 defines a set of data which is used as a case of reference.

Note that for this reference case, which corresponds to an experimental situation discussed subsequently, there is no real pulsating system downstream of the attenuator, but a tube of cross-sectional area A_T which is also air-filled and the oscillation is generated by a loudspeaker (see next section).

The ratio r_{PA} of pressure amplitudes (i.e. pressure amplitude in the resonator vessel divided by $\hat{p}$) is obtained with the help of a simple calculation, making use of the relations (4.14)–(4.16),

$$r_{PA} = \frac{\varrho \cdot c^2}{\omega \cdot V \cdot \hat{p}} \cdot ([A_{III} \cdot \hat{u}_{III} + A_I \cdot \hat{u}_I \cdot \cos(\omega \cdot \tau)]^2 + [A_I \cdot \hat{u}_I \cdot \sin(\omega \cdot \tau)]^2)^{1/2}. \quad (6.1)$$

The mean attenuation power of the Helmholtz resonator is

$$W_A = \frac{1}{2} \cdot A_{III} \cdot \hat{u}_{III} \cdot \hat{p} \cdot \cos(\omega \cdot t_0), \quad (6.2)$$

and the power of a running wave in a cylindric tube of cross-sectional area A_T , which produces a pressure amplitude $\hat{p}$ upon reflection at a rigid

Table 6.1
Data set defining the reference case

pressure amplitude	$\hat{p} = 2.1 \text{ mbar}$
resonance frequency according to equation 2.13	$\omega_0 = 2 \cdot \pi \cdot 217.5 \text{ Hz}$
mean velocity in attenuator tube	$U_{III} = 0; 4 \text{ m/s}$
mean pressure	$\langle p_T \rangle = 0.97 \text{ bar}$
air temperature	$T = 293 \text{ K}$
loss coefficient of inlet tube	$\zeta_I = 2.5$
length of inlet tube	$L_I = 100 \text{ mm}$
diameter of inlet tube	$D_I = 2.8 \text{ mm}$
volume of attenuator vessel	$V = 0.038 \text{ l}$
loss coefficient of attenuator tube	$\zeta_{III} = 0.4$
length of attenuator tube	$L_{III} = 100 \text{ mm}$
diameter of attenuator tube	$D_{III} = 9 \text{ mm}$
length of pulsation tube	$L_T = 1200 \text{ mm}$
cross-sectional area of pulsation tube	$A_T = 0.005027 \text{ m}^2$
gas temperature in pulsation tube	$T_T = 298 \text{ K}$

endwall, is

$$W_T = \frac{A_T \cdot \hat{p}^2}{8 \cdot \varrho_T \cdot c_T}, \quad (6.3)$$

where ϱ_T and c_T denote density and sound speed of the gas in the pulsation tube. Thus, it is assumed that the amplitude of the pressure oscillation exhibits a maximum at the rigid endwall.

The relative attenuation of the Helmholtz resonator is defined to be

$$r_A = \frac{W_A}{W_T}. \quad (6.4)$$

Figures 6-1–6.4 show r_{PA} , r_A , $\hat{u}_{III}$ and $(\omega \cdot t_0)$ as functions of the frequency $\omega/2\pi$ for the data set defined by Table 6.1.

A comparison of the results for the cases with and without flushing confirms the qualitative statements at the beginning of this section.

To illustrate the influence of the thermoacoustic boundary layers in the tube elements of the resonator $\hat{p}$ and $\langle \hat{p}_T \rangle$ are increased by the factor 10. As a consequence ν becomes 10 times smaller and s_I and s_{III} are decreased by the factor $\sqrt{10}$. Comparing the results in Figs. 6.1 and 6.2 shows that for weaker boundary layer effects the ratio of amplitudes and the relative attenuation are increased. Furthermore, the plot of the phase angle $\omega \cdot t_0$ shows (see Fig. 6.4) that the resonance frequency is also increased.

The loss coefficients ζ_I and ζ_{III} deserve our special attention. Increasing ζ_{III} by the factor 2 with respect to the case having $U_{III} = 4$ m/s, $\langle \hat{p}_T \rangle = 0.97$ bar leads to a reduction of r_{PA} of about 20% and a reduction of the

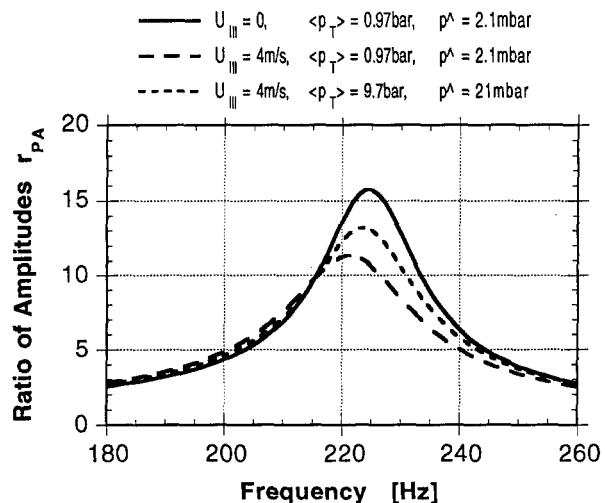


Figure 6.1

Ratio of amplitudes as a function of frequency at different flushing velocities and pressure levels.

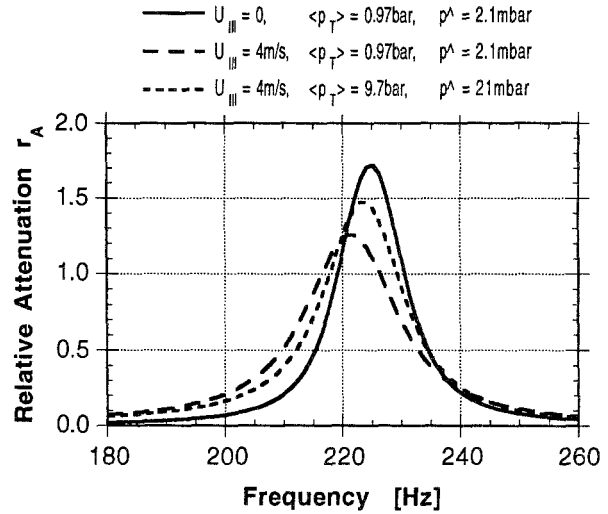


Figure 6.2

Relative attenuation as a function of frequency at different flushing velocities and pressure levels.

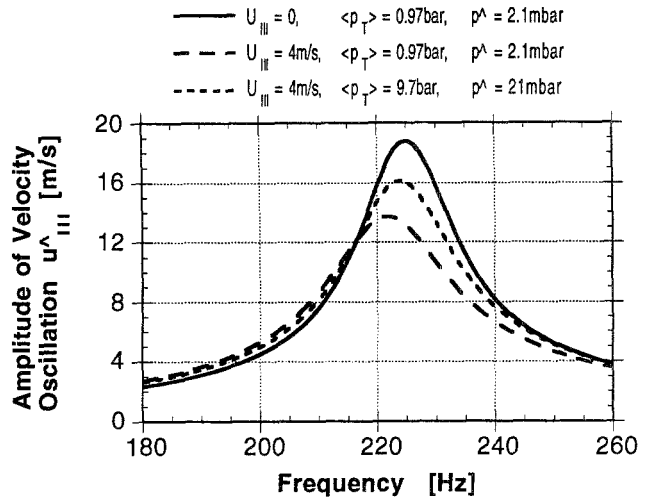


Figure 6.3

Amplitude of velocity oscillation in the attenuation tube as a function of frequency at different flushing velocities and pressure levels.

resonance frequency of a little more than 1%. Dividing ζ_{III} by 2 has the opposite effect.

Increasing ζ_I and keeping the pressure drop across the resonator and the mass flow constant, which implies that A_I has to be increased correspondingly, leads to a strong reduction of r_{PA} and also to a strong increase of the resonance frequency.

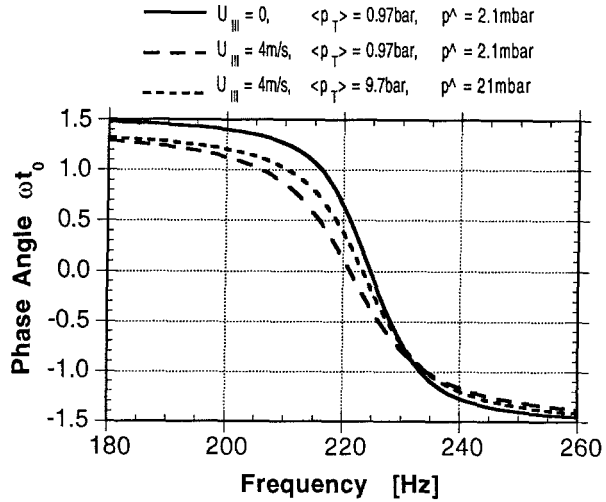


Figure 6.4

Phase angle as a function of frequency at different flushing velocities and pressure levels.

Hence, it is quite important to keep both loss coefficients as small as possible. The values of ζ_{III} are particularly critical. By choosing rounded tube ends (see Fig. 1.1) flow separation in the tube element *III* can be largely suppressed (see Sturtevant and Keller [12]), provided the following Strouhal number criterion is satisfied

$$O\left(\frac{\omega \cdot R_{III}}{2\pi \cdot \dot{u}_{III}}\right) \gtrsim \frac{1}{2}, \quad (6.5)$$

where R_{III} is the radius of curvature of the rounded tube end.

A last aspect which deserves a few additional remarks is associated with the nonlinearity of Helmholtz resonators. As a consequence of the nonlinearity (see e.g. equations (5.14) and (5.15))

$$\dot{u}_{III} \sim \hat{p}^{1/2} \quad (6.6)$$

at resonance, provided the effects of the boundary layers remain weak. Hence, according to equation (6.2)

$$W_A \sim \hat{p}^{3/2}. \quad (6.7)$$

On the other hand, according to equation (5.3),

$$W_T \sim \hat{p}^2. \quad (6.8)$$

Thus, we obtain from (6.4), making use of (6.7) and (6.8),

$$r_A \sim \frac{1}{\hat{p}^{1/2}}, \quad (6.9)$$

which implies that the relative attenuation at resonance becomes weaker with increasing amplitude of the pressure oscillation in the pulsation tube. For this reason, the attenuators should be designed with respect to the largest amplitudes of the pressure oscillations expected in the pulsation tube or in a combustion facility.

7. Experiments

The goals of the experimental investigation are the verification of the analytical model, the acquisition of values for a priori unknown parameters like the loss coefficients and the demonstration of the sensitivity on flushing and on the shape of the throat ends. This was done in a model attenuator using harmonic single frequency excitation.

7.1. Experimental setup

The pulsations are generated in a tube which is equipped with a loudspeaker on one end and a Helmholtz resonator on the opposite end (Fig. 7.1). The dimensions are given in Table 6.1. Large pulsation amplitudes can be achieved if the resonance frequencies of the tube and the resonator coincide. For an arrangement with an open and a closed end the tube length has to be $\lambda/4$, $3\lambda/4$, $5\lambda/4$, $\dots$. The resonator is connected to an air plenum which supplies the flushing air. The air flow is determined by measuring the pressure difference between plenum and tube as well as the temperature in the plenum.

Microphones are used to detect the pulsations in the tube and in the resonator. They are flush mounted in the tube wall and they are embedded in absorbing materials (e.g. foamed material) in order to avoid solid body sound transmission.

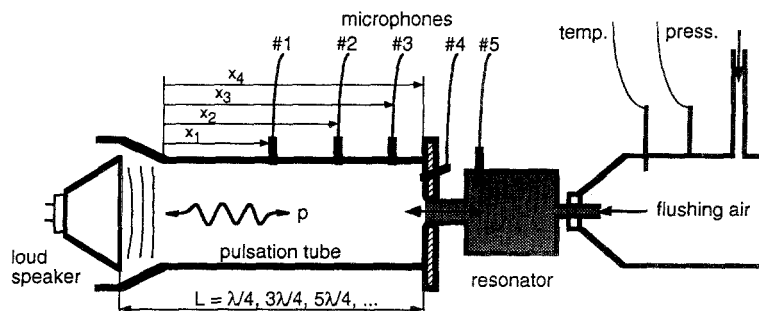


Figure 7.1
Helmholtz attenuator test rig.

The performance of the resonator is determined by two different methods. For the first one the pulsations are measured at two of the positions x_1 , x_2 or x_3 in the tube (Fig. 7.1). For description we choose the positions x_1 and x_2 :

$$\Delta p_1 = A e^{i(\omega t - \Phi_1)} = \frac{1}{2} \varrho [f(x_1) - g(x_1)]; \quad (7.1)$$

$$\Delta p_2 = B e^{i(\omega t - \Phi_2)} = \frac{1}{2} \varrho [f(x_2) - g(x_2)]. \quad (7.2)$$

f and g are the upstream and downstream propagating waves:

$$f = a \exp \left\{ i \left[\omega \left(t - \frac{x}{c(1-M)} \right) + \alpha \right] \right\} \quad (7.3)$$

$$g = b \exp \left\{ i \left[\omega \left(t + \frac{x}{c(1+M)} \right) + \beta \right] \right\}; \quad (7.4)$$

with amplitudes a , b and phase angles α , β .

Having the amplitudes A , B and the phase difference $\Phi_2 - \Phi_1$ of the pulsations the real part of the reflection coefficient can be calculated which is the ratio of the amplitudes of the downstream (reflected) and the upstream propagating (arriving) waves g , f :

$$r^2 = \left(\frac{b}{a} \right)^2 = \frac{1 + \left(\frac{B}{A} \right)^2 - 2 \frac{B}{A} \cos \left[(\Phi_2 - \Phi_1) - \frac{\omega(x_2 - x_1)}{c(1-M)} \right]}{1 + \left(\frac{B}{A} \right)^2 - 2 \frac{B}{A} \cos \left[(\Phi_2 - \Phi_1) + \frac{\omega(x_2 - x_1)}{c(1+M)} \right]}. \quad (7.5)$$

M is the Mach number of the flow in the pulsation tube. In the present investigation M was always very low because the axial flow was only due to a small amount of flushing air. Since there is no further source of attenuation in the present arrangement (if we neglect the small contribution of the thermoacoustic boundary layers in the pulsation tube), the reflection coefficient is a direct measure for the attenuation power of the Helmholtz resonator. The attenuation power which is already defined by equation (6.2) can be written in an alternative form in the limit $M \ll 1$, provided the Helmholtz resonator is connected to the pulsation tube at the location of maximum amplitude of the pressure oscillation,

$$W_A = \frac{A_T \varrho_T c_T}{8} (a^2 - b^2) = \frac{A_T \hat{p}^2}{2 \varrho_T c_T} \frac{1-r}{1+r}, \quad (7.6)$$

where

$$\frac{\hat{p}}{\varrho_T c_T} = \frac{a+b}{2} = a \frac{1+r}{2}. \quad (7.7)$$

Using equations (7.6) and (6.3) we obtain a relation between the relative attenuation (equation (6.4)) and the reflection coefficient:

$$r_A = \frac{W_A}{W_T} = 4 \frac{1-r}{1+r}. \quad (7.8)$$

Measuring the pulsations in the resonator and in front of it (microphones # 4, # 5 in Fig. 7.1) is the second method to check the performance of the resonator. The amplitude ratio (equation (6.1)) and the phase difference can be directly compared with the prediction of the analytical model. Furthermore, the attenuation power can be calculated by applying equations (6.2–6.4). This can be used to prove the consistency of both measurements.

As will be shown during the discussion of the experimental results the relative attenuation obtained from both methods is different. The measurements taken from the microphones in the pulsation tube include a small contribution due to the thermoacoustic boundary layer and therefore lead to a higher attenuation. However, the differences are small. This is confirmed by measurements of the reflection coefficient in the closed tube without attenuator. The values are always very close to one.

Two techniques have been used to measure amplitude ratio and phase difference of harmonic signals. The first method uses an oscilloscope where the signals are displayed in x – y mode. Whenever the phase difference deviates from 0 or $n\pi$, which is an indication of attenuation, the signals trace an ellipse on the screen. Evaluating the amplitudes and the intersection lengths of the ellipse with the x , y -axis allows to calculate the phase difference. This method is easy to handle, accurate and provides an impressive visualization of the attenuation effect. Difficulties occur for not purely harmonic or noisy signals.

The second method makes use of a two channel spectral analyzer. The frequency response measurement gives directly the data needed. Because of an averaging option it can also be applied to noisy signals. Good agreement has been found when comparing both methods.

7.2. Experimental results

In a number of experiments the influence of different parameters has been investigated. The parameter variations are summarized in Table 7.1.

Table 7.1
Summary of parameter variations

		Model attenuator
radius of curvature	R_{II} [mm]	0; 2.25; 15
diameter of air inlet tube	D_I [mm]	0; 2.8
length of air inlet tube	L_I [mm]	2; 100
mean velocity in the throat	U_{III} [m/s]	0; 4

Table 7.2
Tests with Helmholtz attenuators

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Table 7.2 contains the full set of experiments. It shows the geometrical data and the main experimental and analytical (computational) results. The computations are fitted to the experiments with respect to resonance frequency, amplitude ratio and/or relative attenuation by choosing appropriate loss coefficients and corrections of the throat length. f_0 listed in Table 7.2 is the resonance frequency according to the Helmholtz formula (2.13), whereby the Levine-Schwinger correction is taken into account to define L_{III} . For resonators having one throat only also the correction due to higher orders (equation 5.16) is applied.

We first discuss experimental and analytical results for the case of a closed resonator (i.e. no inlet for flushing air) with sharp edged throat ends. Figures 7.2 and 7.3 show the amplitude ratio r_{PA} (pressure amplitude in the resonator divided by the pressure amplitude in the tube, cf. equation (6.1)), the phase angle between the two pressure signals and the relative attenuation r_A for experiment No. 1 (E1). The amplitude ratio and the phase angle can almost perfectly be predicted by choosing a loss coefficient of $\zeta_{III} = 1.85$ and a throat length of $L_{III} = 100$ mm which corresponds exactly to the geometrical length. For a sharp edged throat we expect an almost total loss of kinetic energy at outflow and an additional loss at inflow into the throat due to flow separation (i.e. “jet contraction”). Hence the value of the loss coefficient seems to be reasonable.

In a closed resonator the pulsation in the resonator volume and the velocity in the throat exhibit a phase shift of 90° (cf. equation (4.14)). Hence, the phase angle between the pulsation in the tube at the throat exit plane and in the resonator volume is related to ωt_0 by (cf. equations (4.10)

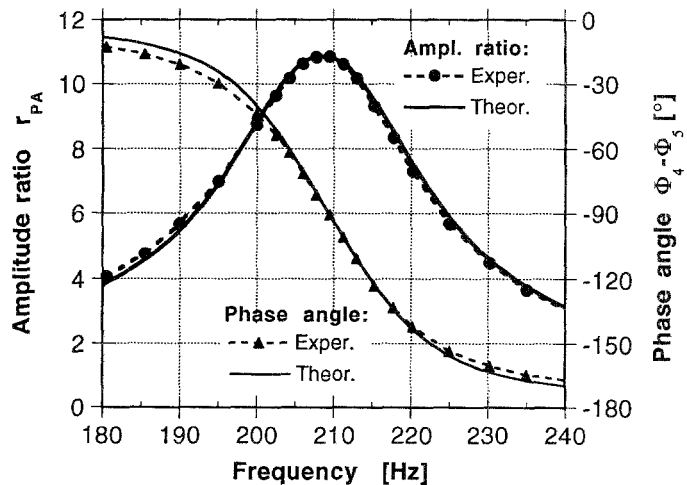


Figure 7.2

Amplitude ratio and phase angle of the closed resonator with sharp edged throat ends (experiment E1).

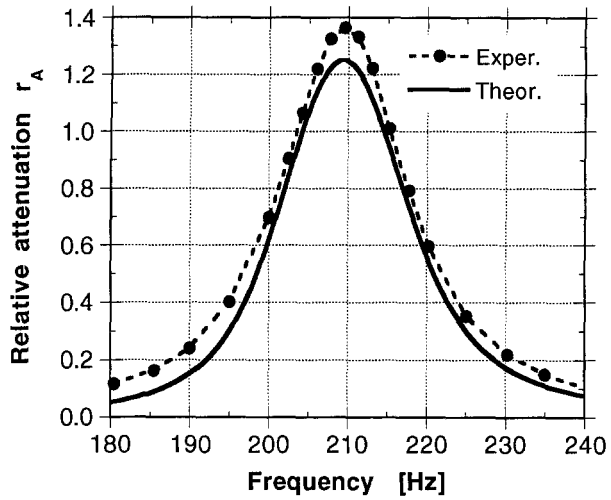


Figure 7.3
Relative attenuation of the closed resonator with sharp edged throat ends (experiment E1).

and (4.15)):

$$\Phi_4 - \Phi_5 = \omega t_0 - 90^\circ. \quad (7.9)$$

At the resonance frequency, $f_r = 209.5$ Hz, the amplitude ratio reaches a maximum of about 11 and the phase shift $\Phi_4 - \Phi_5$ crosses -90° , i.e. $\cos(\omega t_0) = 1$. In other words, the velocity in the throat and the driving pulsation in the tube are in phase which is a necessary condition for maximum attenuation (cf. equation (6.2)). At low frequencies, $f/f_r \ll 1$, the pressure in the tube and in the resonator oscillate in phase while at high frequencies, $f/f_r \gg 1$, these pressures oscillate in counter-phase. Due to a small effect of the boundary layers the values measured for attenuation in the tube are always slightly higher than the computed ones (Fig. 7.3).

The next set of experiments illustrates the influence of flushing on the performance of a Helmholtz resonator which is important for some applications. The relative attenuation of a closed resonator, a resonator with air inlet tube but no flushing and a flushed resonator (E3, E4, E5) is compared in Fig. 7.4. The increase in resonance frequency of about 10 Hz in E4 compared to E3 agrees very well with the Helmholtz formula for a resonator with two throats (equation (2.13)). The peak attenuation is reduced by about 10%. The reason can be seen from equation (A4) by considering the resonant case where $\cos(\omega t_0) = 1$. If a second throat is added the second term on the right-hand side becomes negative because $\sin(\omega \tau) \leq 0$ as defined by equation (A6). A decreasing right-hand side of equation (A4) leads to a reduction of $\hat{u}_{III}$. As a consequence also the attenuation is reduced. Only for an absolutely loss-free air inlet tube ($s_1 = 0$, $\zeta_I = 0$) the performance of the resonator would not deteriorate. In this case $\sin(\omega \tau) = 0$ would be obtained

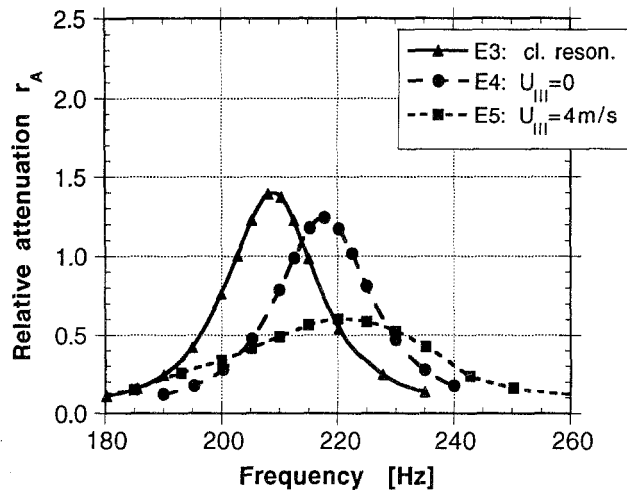


Figure 7.4

Influence of flushing on the relative attenuation of the resonator with sharp edged throat ends (experiments E3, E4, E5).

from equation (A6), i.e. the gas columns in the two throats would be oscillating in counter-phase like the pistons in a boxer engine.

A further reduction in performance occurs if the resonator is flushed as in E5 where $U_{III} = 4\text{ m/s}$. This behaviour appears as a result of the enhanced losses in the throats due to the change of the velocity level, which is accounted for by the function g (equation (5.13), Fig. 5.2). Equation (A4) is directly affected on the left-hand side and indirectly on the right via $\sin(\omega\tau)$, which assumes a larger negative value. The values which are shown in Table 7.2 indicate that flushing shifts $\sin(\omega\tau)$ towards -1 . The pressure in the resonator and the velocity in the inlet tube are then oscillating in phase, i.e. the inlet tube acts like a leak in the resonator.

The experimental results of E3 and E4 can be predicted with only minor changes of ζ_{III} and L_{III} (Table 7.2). In the case of flushing, however, ζ_{III} has to be tripled and L_{III} has to be significantly reduced. The very low Reynolds number of the throat flow and asymmetric flow conditions may serve as an explanation. Errors in the flow rate measurement cannot explain these changes, because U_{III} would have to be increased by the factor 3.5 to reduce the attenuation to the measured value, if the value of ζ_{III} is kept to be 1.85.

Similar experiments as before have been performed for a throat with rounded ends (E6, E7 and E8 shown in Fig. 7.5). An arc of 15 mm radius and an angle of 40° has been used (Fig. 7.6). In this case the peak attenuation increases by more than 55% compared to the sharp-edged throat. The resonance frequency undergoes a shift of 6–7 Hz. The higher values are due to a reduction of the effective length of about 6 mm. The loss coefficients ζ_{III} are decreased to 25–40% of the values for sharp edges.

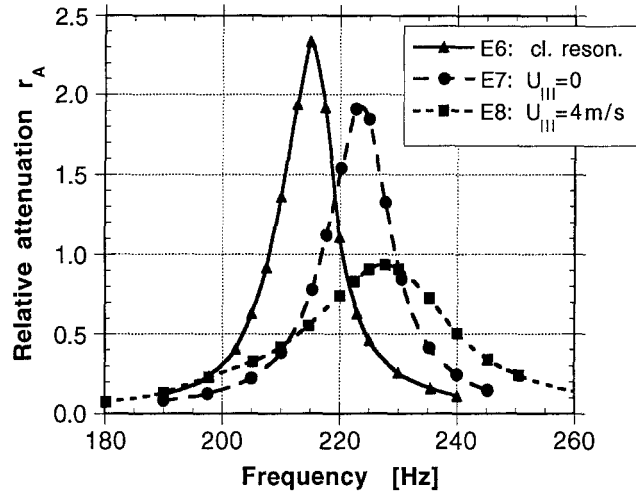


Figure 7.5
Influence of flushing on the relative attenuation of the resonator with rounded throat ends (experiments E6, E7, E8).

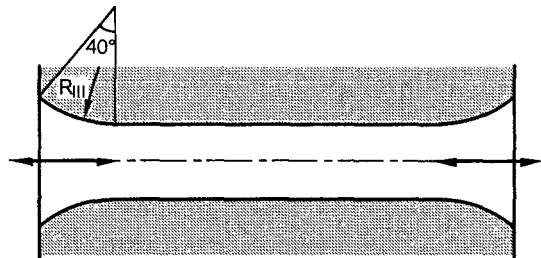


Figure 7.6
Resonator throat with rounded ends used in the experiments E6–E8.

Again the flushed case has an increased loss coefficient although the Strouhal number is about 0.4.

A further experiment (E2) was carried out with a moderate arc radius of 2.25 mm which corresponds to a quarter of the throat diameter. Compared to sharp edges (E1, E3) the loss coefficient is reduced by 40% and the attenuation is increased by 20%.

The experiments E9 and E10 (Table 7.2) show the effect of replacing the air inlet tube ($L_I = 100$ mm) by a simple hole in the resonator ($L_I = 2$ mm). The gas column in the hole has practically no inertia such that $\sin(\omega\tau)$ is very close to -1 . As already pointed out, this is the situation of a leaking resonator. The attenuation is almost completely lost. In this case flushing has some “sealing” effect on the resonator because it reduces the amplitude of the velocity oscillation $\hat{u}_I$ in the hole (cf. equation (A6)).

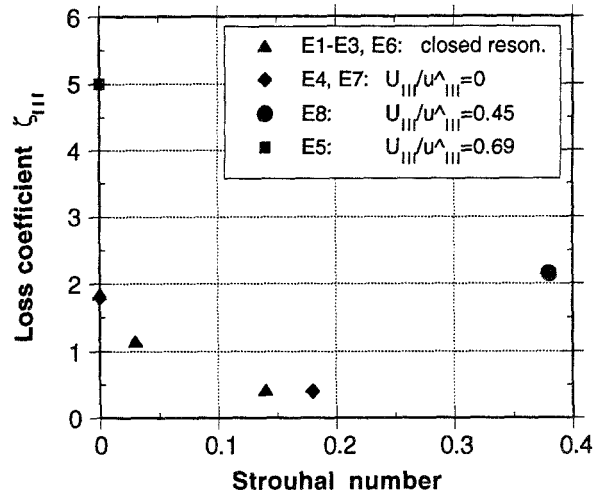


Figure 7.7
Loss coefficient as a function of Strouhal number and flushing velocity (experiments E1–E8).

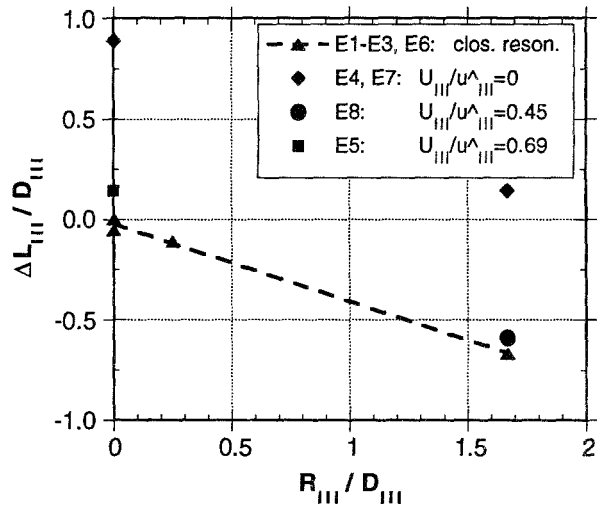


Figure 7.8
Correction of the throat length as a function of rounding radius and flushing velocity (experiments E1–E8).

As already mentioned appropriate loss coefficients and corrections for the throat length have to be chosen in order to fit computational and experimental results (Table 7.2). The values are shown in Figs. 7.7 and 7.8 where the loss coefficient ζ_{III} is displayed versus Strouhal number (equation (6.5)) and the length correction ΔL_{III} is displayed versus radius of curvature R_{III} of the rounded tube ends. Parameter is the flushing velocity U_{III} . Figure 7.7 indicates a strongly reduced loss coefficient even for Strouhal numbers much lower than recommended by the criterion given in equation (6.5).

The throat length (Fig. 7.8) needs no additional correction in case of the closed resonator with sharp edges where the effective length according to equation (5.16) is applied. Rounded tube ends lead to a reduction of the effective throat length as expected. The length correction is about 0.4 times the radius of curvature. For a resonator with two throats equation (5.16) is not valid anymore. As a consequence, the effective length has to be significantly increased in the case without flushing. Flushing, however, reduces the correction which is necessary to match computation and experiment. This points towards strong separation effects because of the asymmetric flow conditions and the reduced unsteadiness of the flow expressed by $U_{III}/\hat{u}_{III} > 0$. The behaviour coincides with the strong increase of the loss coefficient.

8. Conclusions

- (i) The accurate prediction of the frequency-dependent attenuation properties of Helmholtz attenuators requires nonlinear effects due to flow losses, effects due to thermoacoustic boundary layers and the first frequency correction to the “Helmholtz approximation” to be taken into account.
- (ii) To achieve high attenuation performance the flow losses have to be reduced by rounding the ends of the throat and the air inlet tube. So far the loss coefficients and the correction of the throat length have to be determined experimentally.
- (iii) In case of a flushed attenuator the loss of performance can be kept small if the air inlet tube is not too short and if it has a small cross section. A length similar to that of the throat is recommendable.

Appendix

A1:

$$\begin{aligned}
 \frac{\Delta p_6}{\rho c} &= \frac{1}{2} (f_6 - g_6) \\
 &= i \frac{A_{II}}{A_{III}} \cdot \sin\left(\frac{\omega L_I}{c}\right) \cdot \sin\left(\frac{\omega L_{II}}{c}\right) \cdot \sin\left(\frac{\omega L_{III}}{c}\right) \\
 &\quad + \left[\frac{A_{II}}{A_{III}} \zeta_{III} \cdot M_{III} + \frac{A_I}{A_{II}} \cdot \zeta_I \cdot M_I \right] \cdot \sin\left(\frac{\omega L_I}{c}\right) \cdot \sin\left(\frac{\omega L_{II}}{c}\right)
 \end{aligned}$$

$$\begin{aligned}
& \cdot \cos\left(\frac{\omega L_{III}}{c}\right) - i \left[1 + \frac{A_I}{A_{II}} \cdot \frac{A_{II}}{A_{III}} \cdot \zeta_I \cdot M_I \cdot \zeta_{III} \cdot M_{III} \right] \\
& \cdot \sin\left(\frac{\omega L_I}{c}\right) \cdot \cos\left(\frac{\omega L_{II}}{c}\right) \cdot \cos\left(\frac{\omega L_{III}}{c}\right) + \frac{A_I}{A_{II}} \cdot \frac{A_{II}}{A_{III}} \cdot \zeta_I \cdot M_I \\
& \cdot \sin\left(\frac{\omega L_I}{c}\right) \cdot \cos\left(\frac{\omega L_{II}}{c}\right) \cdot \sin\left(\frac{\omega L_{III}}{c}\right) - i \frac{A_I}{A_{II}} \cdot \frac{A_{II}}{A_{III}} \\
& \cdot \cos\left(\frac{\omega L_I}{c}\right) \cdot \cos\left(\frac{\omega L_{II}}{c}\right) \cdot \sin\left(\frac{\omega L_{III}}{c}\right) \\
& - \left[\zeta_I \cdot M_I + \frac{A_I}{A_{II}} \cdot \frac{A_{II}}{A_{III}} \cdot \zeta_{III} \cdot M_{III} \right] \cdot \cos\left(\frac{\omega L_I}{c}\right) \cdot \cos\left(\frac{\omega L_{II}}{c}\right) \\
& \cdot \cos\left(\frac{\omega L_{III}}{c}\right) - i \left[\frac{A_I}{A_{II}} + \frac{A_{II}}{A_{III}} \cdot \zeta_I \cdot M_I \cdot \zeta_{III} \cdot M_{III} \right] \cdot \cos\left(\frac{\omega L_I}{c}\right) \\
& \cdot \sin\left(\frac{\omega L_{II}}{c}\right) \cdot \cos\left(\frac{\omega L_{III}}{c}\right) + \frac{A_{II}}{A_{III}} \cdot \zeta_I \cdot M_I \cdot \cos\left(\frac{\omega L_I}{c}\right) \\
& \cdot \sin\left(\frac{\omega L_{II}}{c}\right) \cdot \sin\left(\frac{\omega L_{III}}{c}\right). \tag{A1}
\end{aligned}$$

A2:

$$\begin{aligned}
& \left[\left(\frac{A_{II}}{A_{III}} \cdot L_I L_{II} L_{III} \cdot \frac{\omega^2}{c^2} \right) - \left(L_I + \frac{A_I}{A_{II}} \cdot \frac{A_{II}}{A_{III}} \cdot L_{III} \right. \right. \\
& \left. \left. + \zeta_I M_I \zeta_{III} M_{III} \cdot \frac{A_{II}}{A_{III}} \cdot L_{II} \right) \right] \\
& \cdot \left[3 \cdot \left(\frac{A_{II}}{A_{III}} \cdot L_I L_{II} L_{III} \cdot \frac{\omega^2}{c^2} \right) - \left(L_I + \frac{A_I}{A_{II}} \cdot \frac{A_{II}}{A_{III}} \cdot L_{III} \right. \right. \\
& \left. \left. + \zeta_I M_I \zeta_{III} M_{III} \cdot \frac{A_{II}}{A_{III}} \cdot L_{II} \right) \right] + 2 \cdot \left(\frac{A_{II}}{A_{III}} \cdot L_{II} \right) \\
& \cdot (\zeta_I M_I L_{III} + \zeta_{III} M_{III} L_I) \cdot \left[\left(\frac{A_{II}}{A_{III}} \cdot L_{II} \right) \right. \\
& \left. \cdot (\zeta_I M_I L_{III} + \zeta_{III} M_{III} L_I) \cdot \frac{\omega^2}{c^2} \right. \\
& \left. - \left(\zeta_I M_I + \zeta_{III} M_{III} \frac{A_I}{A_{II}} \cdot \frac{A_{II}}{A_{III}} \right) \right] = 0. \tag{A2}
\end{aligned}$$

Procedure to solve the system of equations (5.15) and (5.16)

Inserting the fundamental harmonics (4.15) and (4.16) into the equations (5.14) and (5.15) and equating terms of equal phase leads to four equations for the four unknowns $\hat{u}_{III}$, t_0 , $\hat{u}_I$ and τ :

$$\left[\frac{A_{III} \cdot c^2}{V} - (1 + s_{III}) \cdot L_{III} \cdot \omega^2 \right] \cdot \hat{u}_{III} = \frac{\omega \cdot \hat{p}}{\varrho} \cdot \sin(\omega \cdot t_0) - \frac{A_I \cdot c^2}{V} \cdot \cos(\omega \cdot \tau) \cdot \hat{u}_I, \quad (A3)$$

$$\zeta_{III} \cdot \omega \cdot g\left(\frac{U_{III}}{\hat{u}_{III}}\right) \cdot \hat{u}_{III}^2 + s_{III} \cdot L_{III} \cdot \omega^2 \cdot \hat{u}_{III} = \frac{\omega \cdot \hat{p}}{\varrho} \cdot \cos(\omega \cdot t_0) + \frac{A_I \cdot c^2}{V} \cdot \sin(\omega \cdot \tau) \cdot \hat{u}_I, \quad (A4)$$

$$\left[\frac{A_I \cdot c^2}{V} - (1 + s_I) \cdot L_I \cdot \omega^2 \right] \cdot \hat{u}_I = -\frac{A_{III} \cdot c^2}{V} \cdot \cos(\omega \cdot \tau) \cdot \hat{u}_{III}, \quad (A5)$$

$$\zeta_I \cdot \omega \cdot g\left(\frac{U_I}{\hat{u}_I}\right) \cdot \hat{u}_I^2 + s_I \cdot L_I \cdot \omega^2 \cdot \hat{u}_I = -\frac{A_{III} \cdot c^2}{V} \cdot \sin(\omega \cdot \tau) \cdot \hat{u}_{III}. \quad (A6)$$

Knowing the pressure drop across the resonator (see Fig. 1.1) we can determine the mean mass flow and the mean velocities U_I and U_{III} .

Eliminating first $\cos(\omega \cdot \tau)$ from (A3) with the help of (A5) and $\sin(\omega \cdot \tau)$ from (A4) with the help of (A6) and, secondly, eliminating $(\omega \cdot t_0)$ by combining (A3) and (A4) leads to an equation of the form

$$f(\hat{u}_I, \hat{u}_{III}; \text{parameters}) = 0. \quad (A7)$$

Furthermore, eliminating $(\omega \cdot \tau)$ from (A5) and (A6) we obtain an equation of the form

$$g(\hat{u}_I; \text{parameters}) - \hat{u}_{III}^2 = 0. \quad (A8)$$

Finally, eliminating $\hat{u}_{III}$ in (A7) by means of (A8) leads to a single equation for $\hat{u}_I$ which can be solved numerically by looking for the smallest positive zero. Having found $\hat{u}_I$ we obtain $\hat{u}_{III}$ with the help of (A8), $(\omega \cdot \tau)$ with the help of (A5) or (A6) and $(\omega \cdot t_0)$ with the help of (A3) or (A4).

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Summary

In the present paper a nonlinear acoustic theory is proposed, to accurately describe the properties of a generalized type of Helmholtz resonators. The theory may be used as a layout tool to design sound attenuators for combustion facilities. The present investigation shows that, in addition to the nonlinear effects, the effects due to thermoacoustic boundary layers and the first frequency correction to the “Helmholtz approximation” should be taken into account, in order to predict the frequency-dependent attenuation properties of Helmholtz attenuators with the accuracy required by typical technical applications. A series of experiments is presented which is used to validate the theoretical predictions.

Zusammenfassung

In der vorliegenden Abhandlung wird eine nichtlineare akustische Theorie vorgeschlagen, mit der das Verhalten einer verallgemeinerten Art von Helmholtzresonatoren beschrieben werden kann. Die Theorie kann als Werkzeug zur Auslegung von Schalldämpfern für Verbrennungsanlagen benutzt werden. Die vorliegende Arbeit zeigt, dass neben den nichtlinearen Effekten auch die Einflüsse der thermoakustischen Grenzschichten und der ersten Frequenzkorrektur zur “Helmholtzapproximation” berücksichtigt werden müssen, damit die frequenzabhängigen Eigenschaften von Helmholtzdämpfern mit technisch ausreichender Genauigkeit beschrieben werden können. Mit einer Reihe von Experimenten werden die theoretischen Ergebnisse untermauert.

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